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Heat flow rate at a bore-face and temperature in the multi-layer media surrounding a borehole

Vladimir Chugunov [a], Sergei Fomin [b,*], Toshiyuki Hashida [b]

[a] *Department of Applied Mathematics, Kazan State University, Kazan 420008, Russia*
[b] *Fracture Research Institute, School of Engineering, Tohoku University, Sendai 980-8579, Japan*

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Abstract

Assessment of the heat either delivered from high temperature rocks to the borehole or transmitted to the rock formation from circulating fluid is of crucial importance for a number of technological processes related to borehole drilling and exploitation. Normally the temperature fields in the well and surrounding rocks are calculated numerically by finite difference method or analytically by applying the Laplace-transform method. The former approach requires tedious and, in certain cases, non-trivial numerical computations. The latter method leads to rather bulky formulae that are inconvenient for further numerical evaluation. Moreover, in previous studies where the solution is obtained analytically, the heat interaction of the circulating fluid with the formation was treated on the condition of constant bore-face temperature. In the present study the temperature field in the rock formation disturbed by the heat flow from the borehole is modeled by a heat conduction equation, assuming the Newton model for the convective heat transfer on the bore-face, with boundary conditions that account for the thermal history of the borehole exploitation. The problem is solved analytically by the generalized heat balance integral method. Within this method the approximate solution of the heat conduction problem is sought in the form of a finite sum of functions that belong to a complete set of linearly independent functions defined at the finite interval bounded by the radius of thermal influence and that satisfy the homogeneous boundary conditions on the bore-face. In the present study first and second order approximations are obtained for the composite multi-layer domain. The numerical results illustrate that the second approximation is in a good agreement with the exact solution. The only disadvantage of this solution is that it depends on the radius of thermal influence, which is an implicit function of time and can only be found numerically by iterative algorithms. In order to eliminate this complication, in this study an approximate explicit formula for the radius of thermal influence and new close-form approximate solution are proposed on the basis of the approximate solution obtained by the integral-balance method. Employing the non-liner regression method the coefficients for this simplified solution are obtained. The accuracy of the approximate solution is validated by comparison with the exact analytical solution found by Carslaw and Jaeger for the homogeneous domain.

Keywords: Approximate solution; Bore-face; Borehole; Casing; Formation; Heat flux; Temperature

1. Introduction

The heat transfer process plays an important role in a number of industrial applications related to drilling technologies and borehole exploitation. For instance, in geothermal power plants the heat losses from the walls of injection and production wells is an important factor that affects the productivity of the geothermal system [1]. Interpretation of electric logs and estimation of the formation temperatures from well logs also require the knowledge of the temperature disturbances in the formation produced by circulating fluid during drilling or production. Assessment of the heat either delivered

* Corresponding author. Tel.: +81-22-217-7519; fax: +81-22-217-4311.
E-mail address: fomin@rift.mech.tohoku.ac.jp (S. Fomin).

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Nomenclature

A	parameter defined by Eq. (46)
a_k	functions in the approximate solution (8)
Bi	Biot number defined by Eqs. (6)
D	non-dimensional thermal diffusivities defined by Eqs. (7)
D_i	non-dimensional thermal diffusivities, d_i/d_m
d_i	dimensional thermal diffusivity for the ith layer in the domain $(i = 1, 2, \dots, m)$
d_∞	dimensional thermal diffusivity of the rock formation, d_m
$f_k(r)$	functions in the approximate solution (8)
$f'_k(l)$	derivative $\mathrm{d}f_k(r)/\mathrm{d}r$ at $r = l$
H	depth of the borehole
h_w	heat transfer coefficient on the bore-face
J_0, J_1	Bessel functions of the first kind of the order 0 and 1, respectively
q_w^*	dimensional heat flux on the bore-face defined by Eq. (50), which correspond to T^*
$\tilde{q}$, q_w	scaled heat fluxes on the bore-face which correspond to $\widetilde{T}$ and T, respectively
K	thermal conductivities defined by Eq. (7)
K_i	non-dimensional thermal conductivities, k_i/k_1
k_i	dimensional thermal conductivity at the ith layer of the multi-layer domain $(i = 1, 2, \dots, m)$
k_m, k_1	thermal conductivities of the formation and wall of the casing tube, respectively
L	thickness of casing
$l(\tau)$	radius of thermal influence
M_k	functions defined by Eq. (17)
r_w	radius of the borehole
r, z	non-dimensional cylindrical coordinates, r^*/r_w, z^*/H
r_i	non-dimensional bounds of the layers in the domain around the borehole, r_i^*/r_w
T	temperature difference, $T^* - T_0(r^*, z^*, \tau^* + \tau_0)$
T^*	temperature of the media around the borehole
T_0	initial temperature of the rock, $T_0(r^*, z^*, \tau^*)$ which accounts for the thermal history of the formation until time $\tau^* = \tau_0$
T_L^*	mean temperature of the circulating fluid within the borehole, $T_L^*(z^*, \tau^*)$
T_L	temperature defined by Eqs. (6)
$\widetilde{T}$	solution of the problem (1)–(5) for $T_L = 1$
Y_0, Y_1	Bessel functions of the second kind of the order 0 and 1, respectively
v_k	function defined by Eq. (14)
τ	non-dimensional time
τ_0	time at the onset of a new circulation cycle

Superscript

*	dimensional variable

from high temperature rocks to the drilling bit or transmitted to the rock formation from the circulating fluid is of crucial importance for developing new drilling technologies and for optimal design of the instruments used in deep drilling at high formation temperature [2]. Normally the temperature fields in the well and surrounding rocks are calculated numerically [3–8] with a finite difference method. The exact analytical solutions of Shen and Beck [9] and Lee [10] obtained by Laplace transformation are rather bulky and require tedious non-trivial numerical evaluations. Therefore, they are not very convenient for engineering estimation. In a number of previous publications the heat interaction of the circulating fluid with the formation was treated under the condition of constant bore-face temperature [11–17]. On the basis of the latter approach, employing some additional simplifying assumptions, several simple analytical formulae for the temperature in the rock formation and for the heat flux on the bore-face were proposed [14–18]. However, the assumption of constant bore-face temperature is not realistic and, therefore, the temperature on the bore-face should be treated as an unknown function of time and axial coordinate z in mathematical modeling. For this reason, the previously obtained solutions have a limited range of practical applicability and can be used only in the case of highly intensive heat transfer between the circulating fluid and surrounding media. In the present study Newton's model of convective heat transfer on the bore-face is employed. Carslaw and Jaeger [19] found an exact analytical solution of this problem for the particular case of a homogeneous domain, but their solution is rather bulky even for this simple case. The integral-balance approximate method [20,21], applied in the present study, leads to simpler analytical formulae convenient for engineering calculations. It allows solving the heat conduction problem in a composite domain formed with an arbitrary finite number of co-axial cylindrical layers with different thermophysical properties. The existence of the multi-layer composite media around the borehole with different thermophysical properties at each layer occurs due to completion technology that requires, for instance, casing and cementing [6,17].

2. System model and analysis

A schematic sketch of the completed borehole with multi-layer casing is illustrated in Fig. 1. As an example, a 2-layer casing of total thickness L is presented. The axial symmetry of the process is assumed and, therefore, the cylindrical coordinates (r^*, z^*) are employed. In order to focus on the heat conduction problem in the formation and separate this problem from the heat and mass transfer processes in the borehole, the mean fluid temperature in the borehole, $T_L^*(z^*, \tau^*)$, is assumed to be given. The heat flux on the bore-face, $r^* = r_w$, is assumed to be proportional to the temperature difference between the bore-face and the fluid in the well, $-k_1 \partial T^*/\partial r^* = h_w(T_L^* - T^*)$. Obviously, the undisturbed formation temperature $T_f(z^*)$ can be chosen as an initial condition and as the ambient temperature for the mathematical model of the temperature field around the well. However in many applications the circulating regime is not continuous but is often interrupted by shut-in periods. Therefore, the initial temperature $T_0(r^*, z^*, \tau_0)$ at the onset of the specified period $\tau^* = \tau_0$ of fluid circulation in a well should account for the history of the well exploitation. If circulation is continuous or the shut-in periods are sufficiently long, the formation temperature may be restored by the next stage of circulation. In this case it can be assumed that the initial temperature is approximately equal to the undisturbed temperature of the formation, $T_0(r^*, z^*, \tau_0) = T_f(z^*)$. Since radial temperature gradients are much greater than temperature gradients in the vertical direction [10,11], the derivatives with respect to axial coordinate z^* in the governing equations are assumed to be negligibly small.

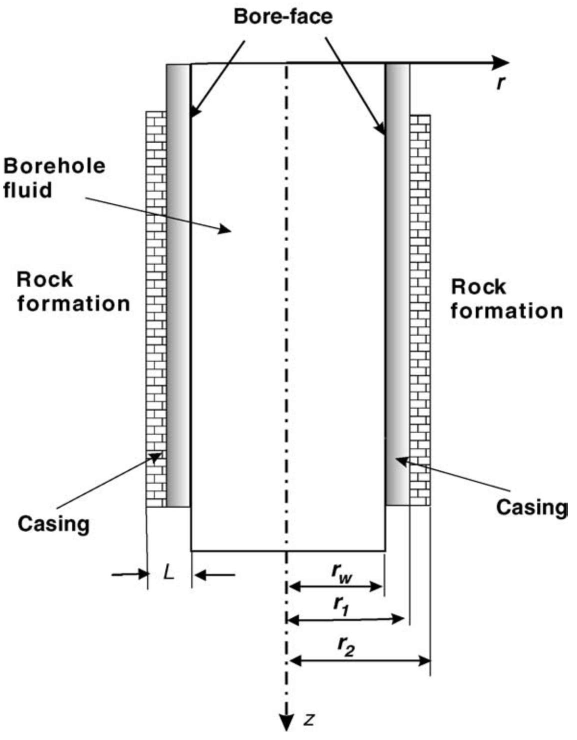

Fig. 1. Schematic sketch of the borehole with multi-layer casing.

2.1. Governing equations

In the case of composite m-layered media around the borehole, with layers bounds determined by the non-dimensional formulae, $r_{i-1} \leqslant r \leqslant r_i$ $(i = 1, 2, \ldots, m;$ $r_0 = 1)$, the temperature distribution in the surrounding rock can be described in cylindrical coordinates (r, z) by the following non-dimensional mathematical model

$$\frac{\partial T}{\partial \tau} = \frac{D(r)}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right), \quad 1 < r < \infty, \ \ \tau > 0; \tag{1}$$

$$\tau = 0, \quad T = 0; \tag{2}$$

$$r = 1, \quad -\frac{\partial T}{\partial r} = Bi[T_L(z, \tau) - T]; \tag{3}$$

$$\lim_{r \to \infty} T < \infty, \tag{4}$$

$$T|_{r=r_i-0} = T|_{r=r_i+0}, \qquad K(r) \frac{\partial T}{\partial r}\bigg|_{r=r_i-0} = K(r) \frac{\partial T}{\partial r}\bigg|_{r=r_i+0} \quad (i = 1, 2, \ldots, m). \tag{5}$$

The non-dimensional quantities in (1)–(5) are introduced by the relationships:

$$\begin{aligned} &r = r^*/r_w, \quad z = z^*/H, \\ &Bi = h_w r_w/k_1, \quad \tau = \tau^* d_\infty/r_w^2, \\ &T = T^*(r^*, z^*, \tau^*) - T_0(r^*, z^*, \tau^* + \tau_0), \\ &T_L(z, \tau) = T_L^* - \left(T_0 - \frac{1}{Bi} \frac{\partial T_0}{\partial r} \right)\bigg|, \end{aligned} \tag{6}$$

where $T_0(r^*, z^*, \tau^* + \tau_0)$ satisfies Eq. (1) and continuity conditions (5).

The coefficients $D(r)$ and $K(r)$ in Eqs. (1)–(5), which characterize the differences in thermophysical properties of the composite multi-layer media surrounding the borehole, can be defined as

$$D(r) = \begin{cases} D_1, & 1 < r < r_1, \\ D_2, & r_1 < r < r_2, \\ \ldots & \ldots \\ D_m, & r_{m-1} < r < \infty, \end{cases} \qquad K(r) = \begin{cases} K_1, & 1 < r < r_1, \\ K_2, & r_1 < r < r_2, \\ \ldots & \ldots \\ K_m, & r_{m-1} < r < \infty, \end{cases} \tag{7}$$

where $D_i = d_i/d_m$, $K_i = k_i/k_1$, and d_i, k_i are the thermal diffusivity and thermal conductivity in the ith layer. For this definition, $D_m = 1$ and $K_1 = 1$. Eq. (1) describes the temperature distribution in the media surrounding the borehole, Eq. (2) is the initial condition at $\tau = 0$, Eq. (3) represents the Newton's law of convective heat transfer at the bore-face, Eq. (4) is the condition of the finite temperature of the rock at infinity, and Eqs. (5) are the continuity conditions at the boundaries of the layers with different thermophysical properties, that constitute the domain surrounding the borehole.

The above formulated mathematical model (1)–(5) can be solved by Laplace transform. However, it will lead to rather awkward formulae. To this end, in this case it would be interesting to employ the approximate integral-balance method [20] for obtaining a simple solution convenient for further incorporation into a complete model of the heat and mass transfer processes during drilling, injection and production periods of borehole exploitation. The analytical approximate method based on integral-heat-balance correlations proposed by Goodman [20] and improved by Volkov et al. [21] is used in the present study for solution of the heat conduction problem (1)–(5) in the composite multilayer domain. This method was successfully applied in [22] to the problem of moving heat source within the borehole in an application involving melting a paraffin deposition in the annulus. The accuracy of the solution found by integral-balance method is validated in the present study by comparison to the exact analytical solution available in [19] for a homogeneous domain with uniform thermophysical properties.

2.2. Description of the generalized integral-balance method

According to [21] the approximate solution of the problem (1)–(5) is sought in the form

$$T(r,\tau) = \begin{cases} T_L + \sum_{k=1}^{n} a_k(\tau) f_k(r), & 1 \leqslant r \leqslant l(\tau), \\ 0, & l(\tau) < r < \infty, \end{cases} \tag{8}$$

where the basis functions f_k (for $k = 1, 2, 3, \dots n$) constitute a complete and linearly independent system for every finite interval $[1, l]$ and satisfy the following relationships:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f_k}{\mathrm{d}r}\right) = f_{k-1}, \quad f_0 = 0, \tag{9}$$

$$r = 1, \quad \mathrm{d}f_k/\mathrm{d}r = Bi \cdot f_k, \tag{10}$$

$$f_k|_{r=r_i-0} = f_k|_{r=r_i+0}, \;\; K(r)\mathrm{d}f_k/\mathrm{d}r|_{r=r_i-0} = K(r)\mathrm{d}f_k/\mathrm{d}r|_{r=r_i+0}, \quad (i = 1, 2, \dots, m;\; k = 1, 2, \dots n). \tag{11}$$

It is approximately assumed that function T and its derivative $\partial T/\partial r$ for each specified moment of time differs from zero only for r within the finite interval $[1, l]$ and at $r = l$ the following conditions for $r = l$ are valid

$$T|_{r=l} = \partial T/\partial r|_{r=l} = 0. \tag{12}$$

So in this sense the function $l(\tau)$ can be referred to as a radius of thermal influence.

Multiplying Eq. (1) by $rf_k(r)$ and integrating on the interval $[1, l]$ while accounting for boundary conditions (2)–(4), yields the recurrent system

$$\frac{\mathrm{d}v_k}{\mathrm{d}\tau} = v_{k-1} + T_L Bi f_k(r)|_{r=1} \quad (v_0 = 0) \tag{13}$$

with initial conditions at $\tau = 0$ of $v_k = 0$, where

$$v_k = \int_1^{l(\tau)} r f_k(r) T(r,\tau)\,\mathrm{d}r. \tag{14}$$

Hence, from Eq. (13) v_k can readily be obtained,

$$v_k = Bi \sum_{j=1}^{k} f_j(1) \underbrace{\int_0^{\tau} \cdots \int_0^{\tau}}_{k-j+1} T_L(\tau)\,\mathrm{d}\tau. \tag{15}$$

On the other hand, substituting (8) into (14), v_k can be presented in the following form

$$v_k = T_L M_k + \sum_{j=1}^{n} a_j(\tau) M_{jk}, \tag{16}$$

where

$$\begin{aligned} M_k(l) &= \int_1^{l(\tau)} r f_k(r)\,\mathrm{d}r, \\ M_{jk}(l) &= \int_1^{l(\tau)} r f_j(r) f_k(r)\,\mathrm{d}r \quad (k, j = 1, 2, \dots). \end{aligned} \tag{17}$$

Combining formulae (15) and (16), and accounting for boundary conditions (12), yields

$$\sum_{j=1}^{n} a_j(\tau) M_{jk} = -T_L M_k + Bi \sum_{j=1}^{k} f_j(1) \underbrace{\int_0^{\tau} \cdots \int_0^{\tau}}_{k-j+1} T_L(\tau)\,\mathrm{d}\tau \quad (k = 1, \dots, n-1), \tag{18}$$

$$\sum_{k=1}^{n} a_k(\tau) f_k(l) = -T_L, \quad \sum_{k=1}^{n} a_k(\tau) f_k'(l) = 0. \tag{19}$$

In Eq. (19) and below the derivatives, $\mathrm{d}f_k/\mathrm{d}r$, at $r = l$ are denoted as $f_k'(l)$ $(k = 1, \dots, n)$. The system of $(n+1)$ Eqs. (18) and (19) is used for calculation of n unknown coefficients $a_k(\tau)$ $(k = 1, 2, \dots, n)$ and of function $l(\tau)$.

For the first approximation $(n = 1)$

$$a_1 = -T_L/f_1(l), \tag{20}$$

and, hence, (8) reduces to

$$T(r,\tau) = T_1(r,\tau) = T_L\left(1 - \frac{f_1(r)}{f_1(l)}\right), \quad 1 \leqslant r \leqslant l(\tau). \tag{21}$$

In this case ($n=1$) the radius of thermal influence is determined by the equation

$$M_1 - \frac{M_{11}(l)}{f_1(l)} = Bif_1(1)\frac{1}{T_L}\int_0^\tau T_L(p)\,\mathrm{d}p. \tag{22}$$

Since, as can be readily shown, $M_{11}(l) = l[f_1(l)f_2'(l) - f_1'(l)f_2(l)]$, and $M_1(l) = lf_2'(l) - Bif_2(1)$, Eq. (22), which defines function $l(\tau)$, can be converted to

$$\frac{f_2(l)}{f_1(l)} - Bif_2(1) = \frac{1}{T_L}\int_0^\tau T_L(p)\,\mathrm{d}p. \tag{23}$$

After a bit more tedious but rather straightforward manipulations, the second approximation ($n=2$) for the temperature distribution in the rocks (8) can be presented by the equation

$$\begin{aligned} T(r,\tau) &= T_2(r,\tau) \\ &= T_L\left(1 - \frac{f_1(r)f_2'(l) - f_2(r)f_1'(l)}{f_1(l)f_2'(l) - f_2(l)f_1'(l)}\right), \quad 1 \leqslant r \leqslant l(\tau), \end{aligned} \tag{24}$$

where l is defined by the equation

$$\begin{aligned} &-Bif_2(1) + \frac{f_1(l)f_3'(l) - f_1'(l)f_3(l)}{f_1(l)f_2'(l) - f_2(l)f_1'(l)}lK(l)f_1'(l) \\ &\quad = \frac{1}{T_L}\int_0^\tau T_L(p)\,\mathrm{d}p. \end{aligned} \tag{25}$$

Accounting for Eqs. (21) and (24), an equation for the heat flux on the bore-face

$$q_w(\tau) = -\frac{1}{Bi}\frac{\partial T}{\partial r}\bigg|_{r=1}$$

can be presented in the following form

$$q_w = T_L(\tau)\tilde{q}, \tag{26}$$

where $\tilde{q}$, obviously, represents the heat flux on the bore-face if in the mathematical model (1)–(5) it is assumed that T_L is constant and equal to unity.

For the first approximation ($n=1$)

$$\tilde{q}(\tau) = \tilde{q}_1(\tau) = \frac{f_1'(1)}{Bif_1(l)}. \tag{27}$$

For the second approximation ($n=2$)

$$\tilde{q}(\tau) = \tilde{q}_2(\tau) = \frac{f_1(1)f_2'(l) - f_2(1)f_1'(l)}{Bi[f_1(l)f_2'(l) - f_2(l)f_1'(l)]}. \tag{28}$$

The basis functions, $f_1(r)$, $f_2(r)$ and $f_3(r)$, that should be substituted into Eqs. (21), (23)–(25), (27) and (28) are defined by the recurrent relationships (9)–(11). Although these functions can be obtained for the heat conduction problem in the composite domain, which contains an arbitrary finite number of layers, $m = 1, 2, 3$, etc., with different thermophysical properties at each layer, in order to avoid bulky formulae, we will consider only the practically important particular case when m is equal to 2. Further analysis of the effect of casing materials with different physical properties on the heat transfer between the rock formation and borehole will be performed on the basis of this simplest model of composite media that is composed of two homogeneous regions. In this case ($m=2$) the formulae for the first three basis functions, f_1, f_2 and f_3 are the following:

$$f_1(r) = \begin{cases} \ln(r) + 1/Bi, & 1 \leqslant r < r_1, \\ \frac{1}{K_2}\ln\frac{r}{r_1} + \alpha, & r_1 \leqslant r < \infty, \end{cases} \tag{29}$$

$$f_2(r) = \begin{cases} \frac{r^2}{4D_1}\left[\ln(r) - 1 + \frac{1}{Bi}\right] + \beta_1, \\ \quad 1 \leqslant r < r_1, \\ \frac{r^2}{4}\left[\frac{1}{K_2}\ln\frac{r}{r_1} + \alpha - \frac{1}{K_2}\right] + \gamma_1\ln\frac{r}{r_1} + \gamma_2, \\ \quad r_1 \leqslant r < \infty, \end{cases} \tag{30}$$

$$f_3(r) = \begin{cases} \frac{r^4}{64D_1^2}\left(\ln r - \frac{3}{2} + \frac{1}{Bi}\right) + \frac{r^2}{4D_1}\beta_1 - \beta_2, \\ \quad 1 \leqslant r < r_1, \\ \frac{r^4}{64}\left(\frac{1}{K_2}\ln\frac{r}{r_1} - \frac{3}{2K_2} + \alpha\right) \\ \quad + \frac{r^2}{4}\left[\gamma_1\left(\ln\frac{r}{r_1} - 1\right) + \gamma_2\right] + \delta_1\ln\frac{r}{r_1} + \delta_2, \\ \quad r_1 \leqslant r < \infty, \end{cases} \tag{31}$$

where

$$\alpha = \ln r + \frac{1}{Bi}, \quad \beta_1 = \frac{Bi^2 - 2Bi + 2}{4Bi^2},$$

$$\beta_2 = \frac{5Bi^3 - 20Bi^2 + 40Bi - 32}{128D_1^2Bi^3},$$

$$\gamma_1 = \frac{r_1^2}{4K_2}\left[2\alpha\left(\frac{1}{D_1} - K_2\right) - \frac{1}{D_1} + 1\right],$$

$$\gamma_2 = \frac{r_1^2}{4}\left[\alpha\left(\frac{1}{D_1} - 1\right) - \frac{1}{D_1} + \frac{1}{K_2}\right] + \beta_1,$$

$$\begin{aligned} \delta_1 &= \frac{r_1^4}{64K_2}\left[4\alpha\left(\frac{1}{D_1^2} - K_2\right) + 5\left(1 - \frac{1}{D_1^2}\right)\right] \\ &\quad + \frac{r_1^2}{4K_2}\left[\frac{2\beta_1}{D_1} + K_2(\gamma_1 - 2\gamma_2)\right], \end{aligned}$$

$$\begin{aligned} \delta_2 &= \frac{r_1^4}{64}\left[\alpha\left(\frac{1}{D_1^2} - 1\right) - 1.5\left(\frac{1}{D_1^2} - \frac{1}{K_2}\right)\right] \\ &\quad + \frac{r_1^2}{4}\left[\frac{\beta_1}{D_1} + (\gamma_1 - \gamma_2)\right] + \beta_2. \end{aligned}$$

As can be seen, the basis functions f_i are simple but rather awkward in the case of a composite domain.

2.3. Solution for the homogeneous domain ($D = K = 1$ and $m = 1$)

If the media surrounding the borehole is homogeneous (i.e. $D = K = 1$, $m = 1$), then in this particular case, from (9)–(11) it follows that

$$f_1(r) = \ln(r) + 1/Bi, \tag{32}$$

$$f_2(r) = \frac{r^2}{4}\left[\ln(r) - 1 + \frac{1}{Bi}\right] + \frac{Bi^2 - 2Bi + 2}{4Bi^2}, \tag{33}$$

$$f_3(r) = \frac{r^4}{64}\left(\ln r - \frac{3}{2} + \frac{1}{Bi}\right) + \frac{r^2}{4}\frac{Bi^2 - 2Bi + 2}{4Bi^2} - \frac{5Bi^3 - 20Bi^2 + 40Bi - 32}{128Bi^3}. \tag{34}$$

Substituting these functions into Eqs. (21)–(28), yields:

For the first approximation,

$$T(r,\tau) = T_L \frac{Bi \ln(l/r)}{1 + Bi \ln l}, \quad 1 \leqslant r \leqslant l(\tau), \tag{35}$$

$$\tilde{q}_1 = 1/(1 + Bi \ln l), \tag{36}$$

where $l(\tau)$ is defined by Eq. (23).

For the second approximation,

$$T(r,\tau) = T_L\left\{1 - \left[\frac{r^2}{4}\left(\ln r + \frac{1}{Bi} - 1\right) + \beta_1 - \frac{l^2}{2}\left(\ln l + \frac{1}{Bi} - \frac{1}{2}\right)\left(\ln r + \frac{1}{Bi}\right)\right] \Big/ \left[\frac{l^2}{4}\left(\ln l + \frac{1}{Bi} - 1\right) + \beta_1 - \frac{l^2}{2}\left(\ln l + \frac{1}{Bi} - \frac{1}{2}\right)\left(\ln l + \frac{1}{Bi}\right)\right]\right\}, \quad 1 \leqslant r \leqslant l(\tau), \tag{37}$$

$$\tilde{q}_2(\tau) = \frac{\frac{2-Bi}{4Bi^2} - \frac{l^2}{2Bi}\left(\ln l + \frac{1}{Bi} - \frac{1}{2}\right)}{\frac{l^2}{4}\left(\ln l + \frac{1}{Bi} - 1\right) + \beta_1 - \frac{l^2}{2}\left(\ln l + \frac{1}{Bi} - \frac{1}{2}\right)\left(\ln l + \frac{1}{Bi}\right)}, \tag{38}$$

where $l(\tau)$ is defined by Eq. (25).

The above solutions for the homogeneous domain are simple and, as it will be shown by comparison with exact solutions, are accurate enough for assessing the temperature field in the rocks and the heat flux on the bore-face. Furthermore, the approximate solutions found for $n = 1$ and $n = 2$ can be made more precise by obtaining the next approximations, since the integral-balance method converges for $n \to \infty$. Although methodologically and technically this method does not present any difficulties for obtaining the 3rd or 4th approximations, doing so leads to a bit more bulky equations that make algebraic evaluations rather tedious.

2.4. Exact analytical solution

If the domain is homogeneous ($m = 1$) and solution of (1)–(4) for $T_L = 1$ is known (denoted by $\widetilde{T}$), then the solution of the above problem for arbitrary T_L can be presented (due to the Duhamel theorem [19]) in the following form:

$$T = \frac{\partial}{\partial \tau}\int_0^\tau T_L(z,p)\widetilde{T}(r,\tau - p)\,\mathrm{d}p, \tag{39}$$

where according to [19],

$$\widetilde{T}(r,\tau) = 1 - \frac{2Bi}{\pi}\int_0^\infty \frac{\{J_0(pr)[pY_1(p) + BiY_0(p)] - Y_0(pr)[pJ_1(p) + BiJ_0(p)]\}}{p\{[pJ_1(p) + BiJ_0(p)]^2 + [pY_1(p) + BiY_0(p)]^2\}}\mathrm{e}^{-p^2\tau}\,\mathrm{d}p. \tag{40}$$

In (40) J_0 and J_1 are Bessel functions of the first kind of the order 0 and 1, respectively; and Y_0 and Y_1 are Bessel functions of the second kind of the order 0 and 1, respectively. Denoting the heat flux on the wall of the well for temperature $\widetilde{T}$ as

$$\tilde{q}(\tau) = -\frac{1}{Bi}\frac{\partial \widetilde{T}}{\partial r}\bigg|_{r=1}$$

and differentiating Eq. (40), yields

$$\tilde{q}(\tau) = \frac{4Bi}{\pi^2} \times \int_0^\infty \frac{\mathrm{e}^{-p^2\tau}\,\mathrm{d}p}{\{[pJ_1(p) + BiJ_0(p)]^2 + [pY_1(p) + BiY_0(p)]^2\}p}. \tag{41}$$

Combining (3), (39) and (41) and accounting for $\tilde{q}(0) = 1$, the non-dimensional temperature and heat flux on the well's wall for the general problem (1)–(4) (in terms of temperature T) can be readily computed:

$$T|_{r=1} = -\int_0^\tau T_L(z,p)\frac{\partial}{\partial \tau}\tilde{q}(r,\tau - p)\,\mathrm{d}p, \tag{42}$$

$$q_w = -\frac{1}{Bi}\frac{\partial T}{\partial r}\bigg|_{r=1} = T_L(z,\tau) + \int_0^\tau T_L(z,p)\frac{\partial}{\partial \tau}\tilde{q}(\tau - p)\,\mathrm{d}p. \tag{43}$$

Eqs. (41)–(43) provide fundamentals for conjugating the temperature field in the rock formation with the heat and mass transfer processes in the borehole. These equations are rather awkward and can present certain difficulties for the numerical computations, especially when coupled with equations for heat and mass transport in the well. In this sense, the simpler approximate solutions obtained above look much more attractive.

2.5. Simple approximate formula for the heat flux on the bore-face

It is worthwhile mentioning that function $l(\tau)$ in the integral-heat-balance method is defined implicitly by the non-linear algebraic equations (23) and (25) and, therefore, can only be found numerically. Instead of correcting the derived solutions by finding the next approximations (leading to more complex equations), it would be interesting to use the obtained above approximate formulae for further simplification of the solution without loss of accuracy. For this purpose we approximately assume that

$$l(\tau) \approx 1 + A(Bi)\sqrt{\tau}, \tag{44}$$

which correctly (due to Eqs. (23) and (25)) simulates the asymptotic behavior of $l(\tau)$ for $\tau \to \infty$ and satisfies the initial condition at $\tau = 0$, $l(0) = 1$. Then on the basis of Eq. (36) it can be suggested that roughly

$$\tilde{q}(\tau) \approx 1/\{1 + Bi \ln[1 + A(Bi)\sqrt{\tau}]\}, \tag{45}$$

where A is an unknown parameter that can be obtained numerically by comparison of the proposed formula (45) with the exact solution (41). Minimizing the squared residual between these two solutions for different Bi, the variation of A with respect to parameter Bi can be readily determined. Suggesting that the approximation for the function $A = A(Bi)$ is in the form of a ratio of linear functions of Bi, the following equation is obtained:

$$A(Bi) = \frac{(2.084 + 0.704Bi)}{(1.554 + 0.407Bi)}. \tag{46}$$

The numerical values for the coefficients in (46) are found by the non-linear regression method where the estimates of the parameters are chosen to minimize the merit function given by the sum of squared residuals.

On the basis of the above formulae, accounting for the structure of Eq. (29), a similar approximate formula for the multi-layer domain can be suggested. For instance, in the case of two regions with different thermophysical properties ($m = 2$) this formula can be presented as:

$$\tilde{q}(\tau) \approx \begin{cases} 1/\{1 + Bi \ln[1 + A(Bi)\sqrt{\tau}]\}, \\ \quad 0 \leqslant \tau < \tau_1, \\ 1/\{1 + Bi \ln(r_1) + Bi \ln[(1 + A(Bi)\sqrt{\tau})/r_1]/K_2\}, \\ \quad \tau_1 \leqslant \tau < \infty, \end{cases} \tag{47}$$

where τ_1 is obtained from the equation $1 + A(Bi)\sqrt{\tau_1} = r_1$ and A is defined by Eq. (46).

Apparently, formula (47) can be readily extended for an arbitrary number of layers m:

$$\tilde{q}(\tau) \approx \begin{cases} 1/\{1 + Bi \ln[1 + A(Bi)\sqrt{\tau}]\}, & 0 \leqslant \tau < \tau_1, \\ 1/\{1 + Bi \ln(r_1) + Bi \ln[(1 + A(Bi)\sqrt{\tau})/r_1]/K_2\}, & \tau_1 \leqslant \tau < \tau_2, \\ \ldots & \ldots \\ 1/\left\{1 + Bi \sum_{j=1}^{m-1} \frac{1}{K_j} \ln\left(\frac{r_j}{r_{j-1}}\right) + Bi \ln[(1 + A(Bi)\sqrt{\tau})/r_{m-1}]/K_m\right\}, & \tau_{m-1} \leqslant \tau < \infty, \end{cases} \tag{48}$$

where $m = 1, 2, 3$, etc., and $r_0 = 1$.

3. Numerical results and discussion

Fig. 2 illustrates the convergence of the approximate integral-balance method. Computations are made for the solution for a homogeneous domain with uniform thermophysical properties. Both approximate solutions (first ($n = 1$) and second ($n = 2$) approximations), represented by the dotted and dashed curves in Fig. 2, are in good agreement with the exact solution (41) (solid curves), though the second approximation exhibits better performance for all values of τ and Bi. For relatively high Biot numbers (enhanced heat transfer and higher fluid flow rates in a borehole) all plots practically merge into one. With reduction of the Biot numbers the discrepancy between the exact and approximate solutions increases. Obtaining the next approximations in terms of the integral-balance method can easily reduce the discrepancy, though doing so leads to more bulky solutions. In general, the accuracy of the second approximation is quite sufficient and, hence, it definitely can be used for calculation of the heat flux on the bore-face and

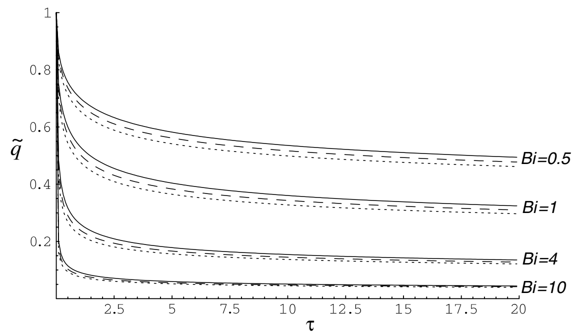

Fig. 2. Variation of the bore-face heat flux $\tilde{q}$ for different Bi. Solid line represents the exact solution (41), dotted line—first approximation (29), (33), and dashed line—second approximation (31), (34).

of the temperature in the rock formation even for low intensity heat transfer between the fluid and formation.

A series of computations was conducted in order to estimate the effect of borehole cementation and casing on the heat flow rate through the bore-face for different flow regimes and casing materials. The thermophysical properties of the casing materials selected for the computations are the following: $r_w = 0.072$ m, $r_1^* = 0.082$ m, $d_1 = 15 \times 10^{-6}$ m^2/s, $k_1 = 46$ W/m K—steel casing; $r_w = 0.072$ m, $r_1^* = 0.082$ m, $d_1 = 0.089 \times 10^{-6}$ m^2/s, $k_1 = 0.115$ W/m K—polyethylene casing; and $r_w = 0.072$ m, $r_1^* = 0.142$ m, $d_1 = 0.46 \times 10^{-6}$ m^2/s, $k_1 = 1.2$ W/m K—for the casing layer of the cement on the walls of the borehole. The heat transfer coefficients for the solid–liquid interface depend on the flow rate and are taken as: $h_w = h_1 = 11$ W/m^2 K and $h_w = h_2 = 110$ W/m^2 K in the numerical computations. For the rocks that constitute the formation, $k_2 = 2.3$ W/m K and $d_2 = 1.2 \times 10^{-6}$ m^2/s. According to the definition (7), in all cases $K_1 = D_2 = 1$. The numerical values of the non-dimensional parameters, which correspond to the above data, are collected in Table 1. The results of the computations for the parameters from the Table 1, presented in Fig. 3, indicate that the thermophysical properties of the casing materials may strongly affect the heat transfer between the flow in the borehole and the surrounding media. The higher values of heat flux correspond to materials with higher thermal conductivity, whereas materials with good insulating properties (like polyethylene) strongly reduce the heat flux on the bore-face. This effect is less pronounced for the higher Biot numbers (higher fluid flow rates in the borehole). The difference between the results for the highly thermally conductive thin casing (steel tubes) and the borehole without casing is relatively small and does not even exceed several percent of the total heat flux. Hence, in this case instead of solving the non-steady heat conduction problem for a multi-layer environment, it is admissible to use the corrected value of the Biot number that accounts for the thermal resistance of the steel casing. Namely, for this approximation, which is widely used in engineering literature [17], the modified Biot number is given by $Bi_{\mathrm{tot}} = Bi/(1 + h_w\delta_c/k_c)$, where δ_c and k_c are the thickness and thermal conductivity of the casing, respectively. It is worthwhile mentioning that this formula should be used with caution. For instance, it cannot be used at the highly unsteady stage of the process at the onset of circulation or for materials of low thermal conductivity. For the latter case, the solution of the unsteady heat conduction problem found in this study for multi-layer arrangements (1)–(5) should be used for the heat transfer analysis.

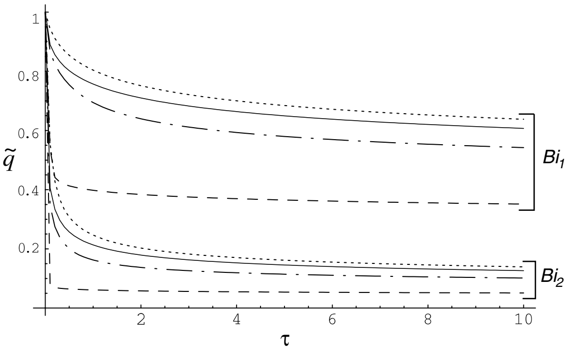

Fig. 3. Heat flux on the bore-face for different casing materials (data from Table 1). (1) Dotted line—steel; (2) Dashed line—polyethylene; (2) Dot-dash line—cemented layer; (4) Solid line—no casing.

As it was mentioned above, one of the disadvantages of the generalized integral-balance method is that the radius of thermal influence $l(\tau)$ can only be obtained numerically by solving Eq. (23) or (25). In order to avoid this complication, the approximate explicit solution (44)–(46) is proposed. Numerical calculations that validate this solution are illustrated in Figs. 4 and 5 where the solid lines correspond to the exact solution for the heat flux on the bore-face (41), the dashed lines to the approximate solution (44)–(46), and the dotted lines to the approximate solutions obtained by the integral-balance method (Eqs. (38) and (25)—second approximation). As can be seen, for all parameters Bi and all times τ the exact solution (solid curves) practically coincides with the data that correspond to the proposed approximate formula (45) (dashed curves in Figs. 3 and 4). The latter justifies the accuracy of the proposed simple solution and proves its applicability to the assessment of the thermal interaction between the borehole and the formation for processes where the thermal history of the formation and borehole axial temperature variation cannot be ignored. For instance, due to Eqs. (6), (35), (36), (44) and (46), if the initial temperature of the formation and of the circulating fluid are known, the

Table 1
Non-dimensional parameters for different casing materials

Type of casing	r_1	K_2	D_1	Bi_1	Bi_2
1. Steel tube	1.2	0.05	12.5	0.018	0.18
2. Polyethylene tube	1.2	20	0.075	7	70
3. Cemented layer	2	1.92	0.38	0.67	6.7
4. No casing	1	$K_1 = K_2 = 1$	$D_1 = D_2 = 1$	0.35	3.5

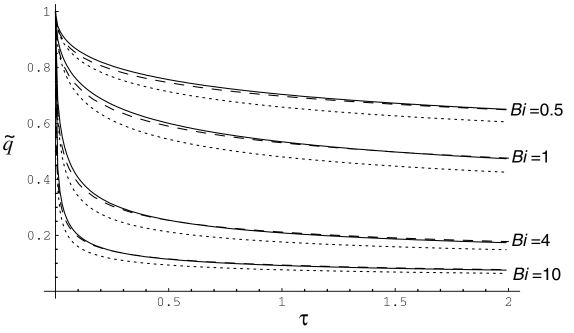

Fig. 4. Short-term ($0 < Fo < 2$) variation of the bore-face heat flux $\tilde{q}$ for different Bi. Solid line represents the exact solution (41), dotted line—approximate solution (29), (33), and dashed line—proposed solution (35), (36).

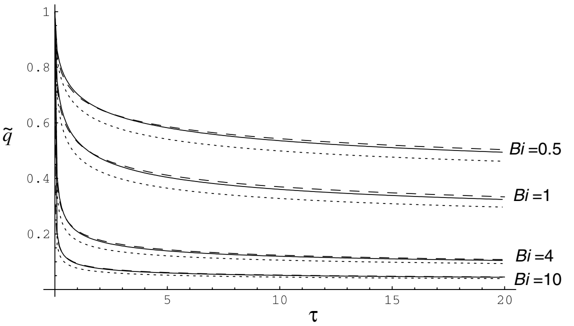

Fig. 5. Long-term ($0 < Fo < 20$) variation of the bore-face heat flux $\tilde{q}$ for different Bi. Solid line represents the exact solution (41), dotted line—approximate solution (29), (33), and dashed line—proposed solution (35), (36).

actual temperature field in the formation T^* and heat flux on the bore-face q^* can be approximated with the following equations

$$T^* = T_0(r,z,\tau+\tau_0) + \left[T_L^*(z,\tau) - \left(T_0 - \frac{1}{Bi}\frac{\partial T_0}{\partial r}\right)\Big|_{r=1}\right] \times \frac{Bi\ln[(1+A(Bi)\sqrt{\tau})/r]}{1+Bi\ln[1+A(Bi)\sqrt{\tau}]}, \tag{49}$$

$$q_w^* = -k_1\left(\frac{\partial T^*}{\partial r}\right)\Big|_{r=1} = -k_1\left(\frac{\partial T_0}{\partial r}\right)\Big|_{r=1} + \frac{k_1 Bi\left[T_L^*(z,\tau) - \left(T_0 - \frac{1}{Bi}\frac{\partial T_0}{\partial r}\right)\big|_{r=1}\right]}{1+Bi\ln[1+A(Bi)\sqrt{\tau}]}, \tag{50}$$

where A is defined by Eq. (46).

In the case of composite (multi-layer) media, provided that the exact close-form solution is not available, the proposed simplified solution (47) can be validated by comparison to the solution obtained by the integral-balance method, which is defined to the second approximation by Eqs. (25), (28)–(31). It is rewarding to see in Fig. 6 that results of numerical computations based on these formulae (dashed and dot-dash lines) show a striking consistency with the approximate solution (47) (dotted lines) for all Bi and τ, for both polyethylene and for cement casings. This astonishing performance of the proposed formula (47) for the particular case of $m = 2$ allows us to also expect that for the general case of an arbitrary number of layers, constituting the media surrounding the borehole, Eq. (48) can be used for computing the heat flow rate at the bore-face with reliable accuracy.

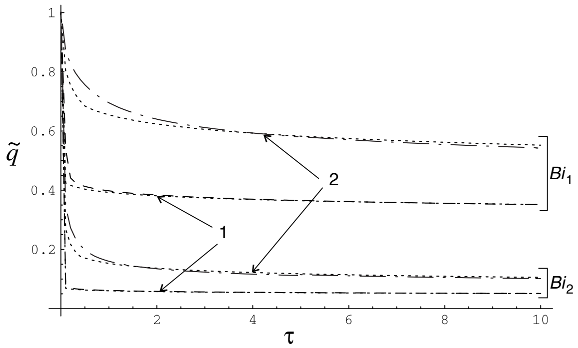

Fig. 6. Heat flux on the bore-face computed with approximate close-form solution (47) (dotted lines) and integral-balance method (Eqs. (25) and (38)) for cement and polyethylene casing materials (data from Table 1). (1) Dashed line—polyethylene casing; (2) Dot-dash line—cemented layer.

4. Conclusions

The following conclusions are drawn:

1. The generalized integral-heat-balance method can be applied to the heat conduction in the multi-layer media surrounding the borehole.
2. The second approximation of the integral-balance method is accurate enough to replace the exact analytical solution found by Laplace transform.
3. The simplified close-form approximate solution for the heat flow rate on the bore-face is proposed. Its accuracy is validated for the homogeneous domain by comparison to the exact solution and for the composite (multi-layer) media by comparison to the solution obtained by the generalized integral-heat-balance method.

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